J. Fluid Mech. (1977), *vol.* 83, *part* 2, *pp.* 289–310
Printed in Great Britain

Differential equations for long-period gravity waves on fluid of rapidly varying depth

By JAMES HAMILTON

Department of Oceanography, University of Southampton, England†

(Received 26 November 1976)

The conventional long-wave equations for waves propagating over fluid of variable depth depend for their formal derivation on a Taylor series expansion of the velocity potential about the bottom. The expansion, however, is not possible if the depth is not an analytic function of the horizontal co-ordinates and it is a necessary condition for its rapid convergence that the depth is also slowly varying. We show that if in the case of two-dimensional motions the undisturbed fluid is first mapped conformally onto a uniform strip, before the Taylor expansion is made, the analytic condition is removed and the approximations implied in the lowest-order equations are much improved.

In the limit of infinitesimal waves of very long period, consideration of the form of the error suggests that by modifying the coefficients of the reformulated equation we may find an equation *exact* for arbitrary depth profiles. We are thus able to calculate the reflexion coefficients for long-period waves incident on a step change in depth and a half-depth barrier. The forms of the coefficients of the exact equation are not simple; however, for these particular cases, comparison with the coefficients of the reformulated long-wave equation suggests that in most cases the latter may be adequate. This opens up the possibility of beginning to study finite amplitude and frequency effects on regions of rapidly varying depth.

1. Introduction

It has long been known that the equations governing the propagation of surface gravity waves can be greatly simplified if the horizontal length scale λ^* (say) of variations in the wave amplitude satisfies

$$\lambda^* \gg h_0, \tag{1.1}$$

where h_0 is a typical depth. In particular Mei & Le Mehauté (1966) have derived to arbitrary order in h_0/λ^* the equations governing the propagation in two dimensions of long waves incident on variable depth.

The presence of bottom topography, however, introduces special problems for the long-wave equations since the interaction of the flow under the wave with the depth must imply that

$$\lambda^* \leqslant \alpha^*, \tag{1.2}$$

† Present address: Department of Applied Mathematics, University of Waterloo, Ontario, Canada.

where α^* is the length scale of variation in the depth. Thus from (1.1) and (1.2) the long-wave equations are valid only for topographies satisfying

$$\alpha^* \gg h_0. \tag{1.3}$$

Inherent in this restriction but also appearing explicitly in the formulation of Mei & Le Mehauté is the condition that all the derivatives of the depth h (say) must exist everywhere. Thus h must be an analytic function of the horizontal co-ordinates. There are formulations, for example Peregrine (1967), which do not include this condition explicitly; however, this is because the equations have not been derived to arbitrary order. Thus the bounds which must be placed on the higher derivatives of h in order for the (absent) neglected terms to be small have not been obtained. Indeed it might be argued that (1.1) and (1.3) provide a fundamental objection to any long-wave type of approach when the depth is rapidly varying and we must consider the problem for a general wavelength. This, however, has proved tractable only for infinitesimal waves (when the problem is linear) and although some solutions for particular topographies are available (Roseau 1952; Bartholomeusz 1968), together with some accurate approximations to global properties, such as reflexion coefficients (e.g. Miles 1967; Mei & Black 1969), the only theory available for reasonably general topographies seems to be that of Kreisel (1949). The restrictions of Kreisel's theory are that the flow be two-dimensional and the depths at either infinity satisfy

$$h(-\infty) = h(+\infty). \tag{1.4}$$

The method, which also requires knowledge of the conformal mapping which takes the fluid domain to a uniform strip, results in an integral equation which must be solved iteratively.

We consider that, because of the restriction (1.4), the greater facility with which differential equations may be handled and the ability of the long-wave equations to handle nonlinear effects, a formulation of the long-wave equations which avoids the analytic restriction and has improved accuracy for rapidly varying depth might be useful in future long-wave investigations.

In the next section we show how for two-dimensional motions and using the conformal-mapping idea of Kreisel we may derive such an equation. However, the consequence of (1.1) and (1.2) is that specific expressions for the error can only be obtained for infinitesimal waves when the frequency δ, say, can be used as a small parameter instead of h_0/λ^*. We shall therefore, for simplicity, consider only infinitesimal waves although nonlinear forms of the equations can be found in appendix A. This ability to include nonlinear effects is of course an important point in favour of this theory as against Kreisel's.

2. Long-wave equations for rapidly varying depth

Derivation of the conventional equations

The equations and boundary conditions governing the propagation of infinitesimal surface gravity waves on fluid of variable depth $h(x)$ take the form (Lamb 1932, p. 363).

$$\phi_{tt} = -g\phi_y \quad \text{on} \quad y = 0, \tag{2.1}$$

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad -h(x) < y < 0, \tag{2.2}$$

$$\phi_x h_x + \phi_y = 0 \quad \text{on} \quad y = -h(x), \tag{2.3}$$

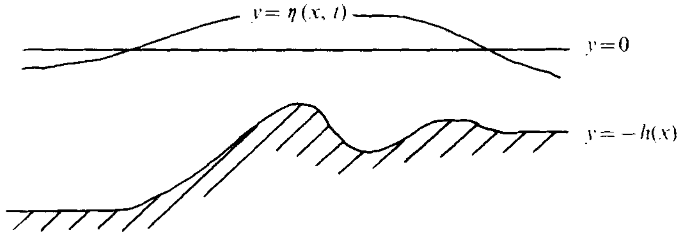

FIGURE 1. Configuration.

where $\phi(x, y, t)$ is the velocity potential of the irrotational flow field, g is the acceleration due to gravity and the configuration is as shown in figure 1. Here and throughout the paper, subscripts generally denote differentiation, x and y are the horizontal and vertical co-ordinates respectively, and t is the time. The free surface

$$y = \eta(x, t) \tag{2.4}$$

may be obtained from the value of the velocity potential on $y = 0$ through the equation

$$\phi_t + g\eta = 0 \quad \text{on} \quad y = 0. \tag{2.5}$$

The fundamental step in obtaining the conventional long-wave equations from (2.1)–(2.3) is the approximation of $\phi(x, y, t)$ by truncations of a Taylor series expansion

$$\phi(x, y, t) = \phi^{(0)}(x, t) + (y + h(x))\,\phi^{(1)}(x, t) + \frac{(y + h(x))^2}{2!}\,\phi^{(2)}(x, t) \ldots \tag{2.6}$$

about the bottom $y + h(x) = 0$ (Mei & Le Mehauté 1966). Substitution of (2.6) into the bottom boundary condition (2.3) gives

$$\phi^{(1)} = -h_x \phi_x^{(0)}/(1 + h_x^2) \tag{2.7}$$

whilst substitution into the continuity equation (2.2) and equating powers of $y + h(x)$ yields the recurrence relation

$$\phi^{(m+2)} = -\{\phi_{xx}^{(m)} + 2h_x \phi_x^{(m+1)} + h_{xx}\phi^{(m+1)}\}/(1 + h_x^2). \tag{2.8}$$

Thus provided that $\phi^{(0)}(x, t)$ and $h(x)$ are analytic functions of x, (2.7) and (2.8) allow the $\phi^{(m)}$ $(m = 1, 2, 3 \ldots)$ to be expressed explicitly in terms of $\phi^{(0)}(x, t) = f(x, t)$, say, $h(x)$ and their derivatives. Substitution of the Taylor expansion (2.6) into the free-surface boundary condition (2.1) then yields a single infinite-order differential equation for $f(x, t)$ which may be truncated under the assumptions (1.1) and (1.3).

For example, the simplest non-trivial truncation of (2.6) takes the form

$$\phi(x, y, t) = f(x, t) - (y + h)\,h_x f_x - \frac{(y + h)^2}{2!} f_{xx} + O\{(h_0/\alpha^*)^4, (h_0/\lambda^*)^4\}, \tag{2.9}$$

where by $O\{(h_0/\alpha^*)^4, (h_0/\lambda^*)^4\}$ we mean a quantity whose order of magnitude relative to the dominant term (here $f(x, t)$) is the *larger* of $(h_0/\alpha^*)^4$ and $(h_0/\lambda^*)^4$. Note that this splitting of the length scales is artificial for rapidly varying depth, however for slowly varying depth this notation allows us to keep better track of the error.

Substitution of (2.9) into the free-surface boundary condition (2.1) gives the well-known equation (e.g. Lamb 1932, p. 273)

$$f_{tt} - \partial[ghf_x]/\partial x = O\{(h_0/\alpha^*)^2, (h_0/\lambda^*)^2\} \tag{2.10}$$

for $f(x, t)$, the value of the velocity potential on the bottom.

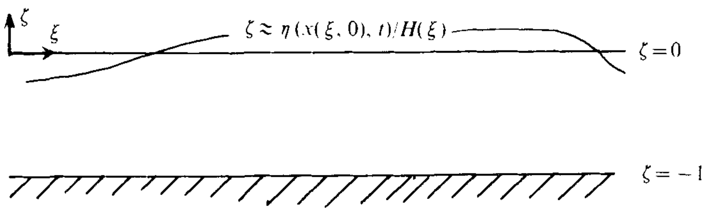

FIGURE 2. Configuration in (ξ, ζ) co-ordinates.

We thus have the not unexpected result that, if the depth $h(x)$ varies on a length scale α^* shorter than the depth, the error will always dominate the terms retained in the conventional long-wave equation. This is so to whatever order (2.6) and (2.10) are taken; indeed the error term in (2.10) is optimistic rather than the reverse since in deriving it we have ignored, for example, $(h_0/\alpha^*)^4$ compared with $(h_0/\alpha^*)^2$. Thus the Taylor series (2.6) for $\phi(x,y,t)$ diverges and a new approach is necessary.

Equations in conformal co-ordinates

We show that by first mapping the (undisturbed) fluid domain onto a uniform strip after the manner of Kreisel (1949) we may improve considerably on the conventional Taylor expansion (2.6) and the resulting equation (2.10). Thus we postulate a conformal mapping

$$\xi + i\zeta = w(x+iy) \tag{2.11}$$

which takes the fluid domain

$$\mathscr{D}: \quad -h(x) < y < 0, \quad -\infty < x < \infty \tag{2.12}$$

to the uniform strip

$$D: \quad -1 < \zeta < 0, \quad -\infty < \xi < \infty \tag{2.13}$$

(see figure 2). The governing equations (2.1)–(2.3) transform in the (ξ,ζ) co-ordinates to (Kreisel 1949)

$$H(\xi)\,\phi_{tt} = -g\phi_\zeta \quad \text{on} \quad \zeta = 0, \tag{2.14}$$

$$\phi_{\xi\xi} + \phi_{\zeta\zeta} = 0 \quad \text{in} \quad -1 < \zeta < 0, \tag{2.15}$$

$$\phi_\zeta = 0 \quad \text{on} \quad \zeta = -1, \tag{2.16}$$

where

$$H(\xi) = Y_\zeta(\xi, 0). \tag{2.17}$$

Note that the influence of the bottom topography has been reduced to there being a smooth (analytic) coefficient $H(\xi)$ in the free-surface boundary conditions. The 'depth' is now constant and we may expand $\phi(\xi,\zeta,t)$ in a Taylor series about the bottom without the problems encountered in the conventional formulation:

$$\phi(\xi,\zeta,t) = \phi^{(0)}(\xi,t) + (\zeta+1)\,\phi^{(1)}(\xi,t) + \frac{(\zeta+1)^2}{2!}\,\phi^{(2)}(\xi,t)\ldots. \tag{2.18}$$

For (2.18) to satisfy the bottom boundary condition (2.16) and the field equation (2.15) we must have

$$\phi^{(1)}(\xi,t) = 0, \quad \phi^{(m+2)}_{\xi\xi} = -\phi^{(m)}_{\xi\xi}. \qquad (2.19), (2.20)$$

Thus we may write (2.18) explicitly in the form

$$\phi(\xi,\zeta,t) = f(\xi,t) - \frac{(\zeta+1)^2}{2!} f_{\xi\xi} + \frac{(\zeta+1)^4}{4!} f_{\xi\xi\xi\xi}\ldots. \tag{2.21}$$

Before going on to write down the various forms of the long-wave equations associated with (2.21) it is as well to consider first the properties of (2.21) as these are significant for the 'finite' amplitude waves as well as for the infinitesimal waves considered here.

Properties of the Taylor expansion

First of all, the radius of convergence of any Taylor expansion of a function is simply the distance to the closest singularity of its associated analytic extension in the complex plane (Phillips 1957). Since ϕ is a solution of Laplace's equation, in this case the radius of convergence is simply the distance to the closest singularity of ϕ or its analytic extension beyond the boundaries of D or $\mathscr{D}$. If the bottom topography possesses corners, these will give singularities in ϕ_x and ϕ_y actually on $y = h(x)$ and thus we are unable to put a lower limit on the radius of convergence of (2.6). In contrast (2.21) is symmetrical about $\zeta+1 = 0$ and thus the closest singularity must be either on or outside the free surface $\zeta = 0$ or its image in the bottom $\zeta = -2$. Assuming that there are no singularities in the surface elevation η and hence in $\phi(\xi, 0, t)$ we have the important result that for infinitesimal waves (2.21) is convergent in

$$|\zeta+1| \leqslant 1. \tag{2.22}$$

For finite amplitude waves, of course, this result must be relaxed somewhat.

The second important advantage of (2.21) as compared with (2.6) is the improvement which results in the lowest-order approximations; this suggests that, as well as converging eventually if sufficient terms are taken, (2.21) will also be significantly more accurate than (2.6) for only a few terms.

The velocities implied by the first terms of (2.6) and (2.21) are respectively

$$\mathbf{q}^{(0)} = \phi_x^{(0)}(x,t)\,\mathbf{i} \tag{2.23}$$

and

$$\mathbf{q}^{(0)} = \phi_\xi^{(0)}(\xi,t)\,\nabla\xi, \tag{2.24}$$

where ∇ is the two-dimensional gradient operator $\mathbf{i}\partial/\partial x+\mathbf{j}\partial/\partial y$. Thus, while (2.23) embodies Lamb's original assumption that the pressure be sensibly hydrostatic and therefore that the flow be parallel to the mean free surface, (2.24) can be interpreted as saying that the velocity field is everywhere parallel to the steady solution of the equations, a much more plausible assumption. Thus the bottom boundary condition is satisfied exactly to whatever order (2.21) is truncated, whilst this is never the case for the conventional expansion (2.6).

The reformulated equations

Substitution of (2.21) into the free-surface boundary condition (2.14) and neglecting $f_{\xi\xi}$ compared with f, etc., gives for first-order balance the equation

$$Hf_{tt} - gf_{\xi\xi} = O\{\lambda^{-2}, \alpha^{-2}\}, \tag{2.25}$$

where λ and α are the (non-dimensional) scales of variation of $f(\xi, t)$ and $H(\xi)$ respectively with the transformed co-ordinate ξ. Again, for consistency in (2.25)

$$\lambda \leqslant \alpha. \tag{2.26}$$

However $H(\xi)$ is an analytic function and in fact we may show (Hamilton 1974; or see appendix B) that

$$H(\xi) = \int_{-\infty}^{\infty} \frac{h(x(\xi^* - 1))\, d\xi^*}{\cosh^2 \frac{1}{2}\pi(\xi^* - \xi)}. \tag{2.27}$$

Thus $\alpha = O(1)$ at the worst and for even the most extreme topographies the neglected terms in (2.25) will never dominate the terms retained, although they may be of the same order of magnitude.

For better comparison of (2.25) with the conventional equation (2.10) we make the transformation

$$x = x(\xi, 0), \quad \xi = \xi(x, 0). \tag{2.28}$$

Thus, noting that the Cauchy–Riemann relations and the definition of $H(\xi)$ give

$$H(\xi) \equiv x_\xi(\xi, 0), \tag{2.29}$$

we obtain

$$\frac{\partial^2 f}{\partial t^2} - g\frac{\partial}{\partial x}\left\{H(\xi(x,0))\frac{\partial f}{\partial x}\right\} = O\{\lambda^{-2}, \alpha^{-2}\}. \tag{2.30}$$

Thus the actual depth $h(x)$ in (2.10) has been replaced by the analytic function

$$H(\xi(x, 0)), \tag{2.31}$$

which we may call the 'equivalent depth'.

Figures 4 and 5 show comparisons of $h(x)$ and the equivalent depth for a step change in depth and a half-depth barrier. It can be seen that, as indicated by (2.27), the equivalent depth can be regarded as a smoothed version of the actual depth $h(x)$.

Now although we have demonstrated that (2.30) is based on more plausible physical assumptions than is the equivalent conventional equation (2.10) and that it is also based on a Taylor expansion for the velocity potential which is convergent for abrupt topography (as opposed to (2.10), which is not), it would still be instructive to obtain a more specific expression for the actual error of (2.30) when $\alpha = O(1)$ than is provided merely by the sum of the remainder terms of the Taylor expansion. This proves possible for the small amplitude waves considered since we may assume that the motions are simple harmonic with a frequency ω, say, satisfying

$$\delta^2 = \omega^2 h_0/g \ll 1, \tag{2.32}$$

where δ is the non-dimensional frequency. Under this assumption (2.25) takes the form

$$f_{\xi\xi} + (\delta^2 H/h_0) f = O\{\delta^2, \alpha^{-2}\}. \tag{2.33}$$

Note that (2.33) should not be taken to mean that the length scale λ of $f(\xi, t)$ is $O(\delta^{-1})$. In fact

$$\lambda = \min(\delta^{-1}, \alpha). \tag{2.34}$$

For example,

$$f_{\xi\xi\xi\xi} \neq O(\delta^4 f) \tag{2.35}$$

since from (2.33)

$$\begin{aligned} f_{\xi\xi\xi\xi} &= (-\delta^2/h_0)(H_{\xi\xi} f + 2H_\xi f_\xi + H f_{\xi\xi}) \\ &= (-\delta^2/h_0)(H_{\xi\xi} f + 2H_\xi f_\xi) + O(\delta^4 f) \\ &= O(\delta^2 \alpha^{-2} f). \end{aligned} \tag{2.36}$$

Note that we have expressed $f_{\xi\xi\xi\xi}$ to $O(\delta^2 f)$ as a linear combination of f and f_ξ and in fact we may do this for all the higher derivatives of f provided that f satisfies (2.33). In general

$$\frac{\partial^{m+2} f}{\partial \xi^{m+2}} = -\frac{\delta^2}{h_0}\left(\frac{\partial^m H}{\partial \xi^m} f + m\frac{\partial^{m-1} H}{\partial \xi^{m-1}} f_\xi\right) + O(\delta^4 \alpha^{-m+2} f). \tag{2.37}$$

Hence we may obtain a modified expression for the error in the free-surface boundary condition when $f(\xi, t)$ is a solution of the truncated equation (2.32):

$$\begin{aligned}\left(\frac{\delta^2 H}{h_0}\phi - \phi_\zeta\right)_{\zeta=0} &= \sum_{m=0}^{\infty} \frac{\delta^2 H}{h_0}\frac{(-1)^{m+1}}{(2m)!}\frac{\partial^{2m} f}{\partial \xi^{2m}} - \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(2m+1)!}\frac{\partial^{2m+2} f}{\partial \xi^{2m+2}} \\ &= \frac{\delta^2 f}{h_0}\left[-H(\xi) - \left[\sum_0^\infty \frac{(-1)^m \zeta^{2m+1}}{(2m+1)!}\frac{d^{2m}x(\xi,0)}{d\xi^{2m}}\right]_{\zeta=-1}\right] \\ &\quad + \frac{\delta^2 f_\xi}{h_0}\left[\sum_{m=0}^{\infty}(-1)^m \frac{\zeta^{2m}}{(2m)!}\frac{d^{2m}x(\xi,0)}{d\xi^{2m}} - \sum_{m=0}^{\infty}\frac{(-1)^m \zeta^{2m+1}}{(2m+1)!}\frac{d^{2m}x(\xi,0)}{d\xi^{2m}}\right]_{\zeta=-1} \\ &= \frac{\delta^2 f}{h_0}[-H(\xi) - y(\xi,-1)] + \frac{\delta^2 f_\xi}{h_0}\left[x(\xi,-1) - \int_{-1}^{0} x(\xi,\zeta)\,d\zeta\right] + O(\delta^4 f).\end{aligned} \tag{2.38}$$

We may interpret the coefficients in the above expression in terms of the stream and potential functions of the steady flow, namely $\zeta(x,y)$ and $\xi(x,y)$. For example

$$(-y(\xi,-1) - H(\xi)) \tag{2.39}$$

is the difference between the depth at the bottom of the line ξ = constant and the value of the equivalent depth function at the top. Similarly

$$x(\xi,-1) - \int_{-1}^{0} x(\xi,\zeta)\,d\zeta) \tag{2.40}$$

is the difference between the horizontal location of the bottom of the line ξ = constant and its 'mean' location. The amount by which solutions of (2.25) fail to satisfy the free-surface boundary condition is thus to $O\{\delta^2\}$ a linear combination of the coefficients (2.39) and (2.40), which are functions only of the topography and the values of f and f_ξ.

More important than the particular forms of (2.39) and (2.40) are, first, the ability to simplify the form of the error in the manner indicated by (2.38), and second, the presence of the term in f_ξ in (2.38). These suggest that we might be able to choose the two coefficients $a(\xi)$ and $b(\xi)$, say, of an equation of the form

$$f_{\xi\xi} + \delta^2[a(\xi) f + b(\xi) f_\xi] = O\{\delta^2\} \tag{2.41}$$

such that the two coefficients of the error term equivalent to (2.39) and (2.40) can be made to vanish identically. The amount by which solutions of (2.41) fail to satisfy the free-surface boundary condition will then be $O(\delta^2)$ relative to ϕ_ζ or $(\delta^2 H/h_0)\phi$ on $\zeta = 0$. Such an equation would correctly describe the dynamics of long-period waves incident on arbitrarily rapidly varying topography. We note that the introduction of the term $\delta^2 b f_\xi$ makes this equation qualitatively different from the simplest form of the reformulated long-wave equations (2.25). However, if we make the assumption $\alpha^{-1} = O(\delta^{\frac{1}{2}})$ and truncate (2.21) in (2.14) accordingly, thus treating the case of the bottom varying more rapidly than the intrinsic length scale of the long waves, we obtain to the lowest order

$$(\delta^2 H/h_0)(f - \tfrac{1}{2} f_{\xi\xi}) + f_{\xi\xi} - \tfrac{1}{6} f_{\xi\xi\xi\xi} = O\{\delta^4, \alpha^{-4}\}. \tag{2.42}$$

Also, since from a first-order balance

$$f_{\xi\xi} = (-\delta^2 H/h_0)f + O\{\delta^2, \alpha^{-2}\}, \tag{2.43}$$

we have on using this to simplify the higher-order terms in (2.42) the equation

$$\frac{\delta^2 Hf}{h_0} + f_{\xi\xi} + \frac{1}{6}\frac{\delta^2}{h_0} H_{\xi\xi} f + \frac{1}{3}\frac{\delta^2}{h_0} H_\xi f_\xi = O\{\delta^2, \alpha^{-4}\}. \tag{2.44}$$

Thus, approximately,

$$a(\xi) = H + \tfrac{1}{6}H_{\xi\xi} + O\{\alpha^{-4}\}, \tag{2.45}$$

$$b(\xi) = \tfrac{1}{3}H_\xi + O\{\alpha^{-2}\}. \tag{2.46}$$

It will be seen that the variable $f(\xi, t)$ is the most natural variable to use for an exact very-long-period equation; it is not in general the most easily observed. Therefore in the next section we derive an equation for the wave amplitude $\eta(\xi, t)$ of the form

$$\eta_{\xi\xi} + \delta^2[A(\xi)\eta + B(\xi)\eta_\xi] = 0. \tag{2.47}$$

The approximate version of this equation equivalent to (2.44) can be found from (2.5), which to order α^{-4} takes the form

$$\eta = (-i\delta/g)[f - \tfrac{1}{2}f_{\xi\xi}] + O\{\alpha^{-4}\}. \tag{2.48}$$

Substitution of (2.48) into (2.44) yields

$$\eta_{\xi\xi} + (\delta^2/h_0)[(H - \tfrac{1}{3}H_{\xi\xi})\eta - \tfrac{2}{3}H_\xi \eta_\xi] = O\{\delta^2, \alpha^{-4}\} \tag{2.49}$$

and the approximations

$$A(\xi) = H - \tfrac{1}{3}H_{\xi\xi} + O\{\alpha^{-4}\}, \tag{2.50}$$

$$B(\xi) = \quad -\tfrac{2}{3}H_\xi \ + O\{\alpha^{-2}\}. \tag{2.51}$$

3. An exact equation for long-period waves

The Taylor expansion (2.21) for the velocity potential ϕ about the bottom is absolutely convergent in $-1 \leqslant \zeta \leqslant 0$ and thus on substitution into the free-surface boundary condition (2.14) we obtain the following infinite-order differential equation for $f(\xi, t) \equiv \phi(\xi, -1, t)$:

$$\frac{\delta^2 H}{h_0}\left\{f - \frac{1}{2!}f_{\xi\xi} + \frac{1}{4!}f_{\xi\xi\xi\xi}\ldots\right\} + \left\{f_{\xi\xi} - \frac{1}{3!}f_{\xi\xi\xi\xi} + \frac{1}{5!}f_{\xi\xi\xi\xi\xi\xi}\ldots\right\} = 0. \tag{3.1}$$

If we assume that $f(\xi, t)$ also satisfies (2.41), which we write in the form

$$f_{\xi\xi} = (-\delta^2/h_0)\{af + bf_\xi\} + O\{\delta^2\}, \tag{3.2}$$

where $a(\xi)$ and $b(\xi)$ are to be determined, then we may write

$$\frac{\partial^{m+2} f}{\partial \xi^{m+2}} = -\frac{\delta^2}{h_0}\frac{\partial^m a}{\partial \xi^m} f - \frac{\delta^2}{h_0}\left[2m\frac{\partial^{m-1} a}{\partial \xi^{m-1}} + \frac{\partial^m b}{\partial \xi^m}\right] f_\xi + O\{\delta^2\}, \quad m = 0, 1, 2 \ldots. \tag{3.3}$$

We find that on substitution for the higher derivatives of f in (3.1) using (3.3) and equating terms in $f(\xi, t)$ and $f_\xi(\xi, t)$ to zero we obtain two infinite-order ordinary

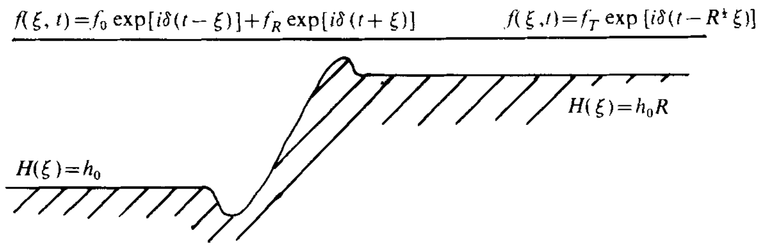

FIGURE 3. Reflexion of a long-period wave by an abrupt change in depth.

differential equations for $a(\xi)$ and $b(\xi)$ as functions of $H(\xi)$. These may be inverted and we quote the results (Hamilton 1974; or see appendix B for the method):

$$a(\xi) = \int_{-\infty}^{\infty} \frac{\pi}{4}\frac{H(\xi^*)}{\cosh^2\frac{1}{2}\pi(\xi^*-\xi)}d\xi^*, \tag{3.4}$$

$$b(\xi) = \int_{-\infty}^{\infty} \frac{\pi}{4}\frac{(\xi^*-\xi)\,H(\xi^*)}{\cosh^2\frac{1}{2}\pi(\xi^*-\xi)}d\xi^*. \tag{3.5}$$

Unfortunately we are primarily interested not in $f(\xi,t)$ but rather in the wave amplitude $\eta(\xi,t)$ and although from (2.5), (2.21) and (3.2) we may obtain

$$\eta(\xi,t) = (-i\delta/g)\,f+O\{\delta^2\} \tag{3.6}$$

$\eta(\xi,t)$ is not itself a solution of (3.2). Given (3.6), however, do we need to derive an equation of the form (2.47) for η? The answer is yes, since (2.47) is the simplest equation which correctly predicts the dynamical behaviour of long-period waves over extensive areas of rapidly varying topography.

Indeed for areas whose extent is very much less than $\delta^{-1}h_0$ we may employ the trivial equation

$$f_{\xi\xi} = O(\delta^2 f) \tag{3.7}$$

to integrate $f(\xi,t)$ across the area of rapidly varying depth, then use the result to splice together two solutions for constant depth. Thus consider figure 3. From (3.7), at the transition ($\xi = 0$ say) both f_ξ and f are continuous, hence

$$f_0-f_R = R^{\frac{1}{2}}f_T+O\{\alpha\delta^2/R\} \tag{3.8}$$

and

$$f_0+f_R = f_T+O\{\alpha^2\delta^2/R^2\}. \tag{3.9}$$

Thus

$$f_R = f_0(1-R^{\frac{1}{2}})/(1+R^{\frac{1}{2}}), \tag{3.10}$$

which reproduces the reflexion coefficients of Lamb (1932, p. 262) and Bartholomeusz (1958) in the limit $\delta\to 0$. Note that f_ξ = constant implies constant mass flux and that the phase lag in the ξ co-ordinate is $O(\alpha\delta/R)$.

Equation (3.7), however, is not sufficient for extensive areas of rapidly varying depth ($\alpha = 1$) and thus all the terms on the right-hand side of (3.2), being of the same magnitude, are important for expressing the dynamical balance correctly. As a direct corollary, components of $f(\xi,t)$ which are rapidly varying (length scales $O(1)$) and therefore from (3.2) have amplitudes $O(\delta^2)$ relative to the components of $f(\xi,t)$ which are slowly varying (length scales $O(\delta^{-1})$) are important in the dynamical balance and should be calculated. Thus the $O(\delta^2)$ terms in (3.6), which are neglected if we calculate $\eta(\xi,t)$ through (3.2) and (3.6), contain rapidly varying components and this method of determining η is inadequate.

We may derive expressions for the coefficient $A(\xi)$ and $B(\xi)$ in an equation of the form

$$\eta_{\xi\xi}+\delta^2(A\eta+B\eta_\xi) = O\{\delta^2\} \tag{3.11}$$

for $\eta(\xi,t)$ by determining the $O\{\delta^2\}$ terms in (3.6) and substituting for η into (3.2). However, the expressions obtained are inconvenient for practical evaluation and we prefer to derive them by expanding the velocity potential about the mean free surface.

Thus the appropriate solution of Laplace's equation (2.15) and the *free-surface* boundary condition (2.14) takes the form

$$\phi = F(\xi,t)-\frac{\zeta^2}{2!}F_{\xi\xi}+\frac{\zeta^4}{4!}F_{\xi\xi\xi\xi}\ldots+\frac{\delta^2}{h_0}\left(\zeta HF-\frac{\zeta^3}{3!}(HF)_{\xi\xi}+\frac{\zeta^5}{5!}(HF)_{\xi\xi\xi\xi}\ldots\right), \tag{3.12}$$

where $F(\xi,t) \equiv \phi(\xi,0,t)$ and the wave amplitude η, from (2.5), is simply proportional to F:

$$\eta = (-i\omega/g)F. \tag{3.13}$$

However, in contrast to expansion of the velocity potential about the free surface (2.21) we have no absolute assurance that (3.12) is convergent in $-1 \leqslant \zeta \leqslant 0$ and the resulting equation (3.11) must be justified through (2.5), (2.21) and (3.2). Thus assuming the convergence of (3.12), we may substitute into the bottom boundary condition, and using (3.13) obtain the following infinite-order ordinary differential equation for η:

$$\eta_{\xi\xi}-\frac{1}{3!}\eta_{\xi\xi\xi\xi}+\frac{1}{5!}\eta_{\xi\xi\xi\xi\xi\xi}\ldots+\frac{\delta^2}{h_0}\left\{H\eta-\frac{1}{2!}(H\eta)_{\xi\xi}+\frac{1}{4!}(H\eta)_{\xi\xi\xi\xi}\ldots\right\} = 0. \tag{3.14}$$

Now, under the assumption that η also satisfies an equation of the form (3.11) where $A(\xi)$ and $B(\xi)$ are as yet undetermined functions of ξ, we know that solutions of (3.11) satisfy

$$\eta_{\xi\xi} = (-\delta^2/h_0)(A\eta+B\eta_\xi), \tag{3.15}$$

and on differentiating with respect to ξ we get

$$\eta_{\xi\xi\xi} = (-\delta^2/h_0)(A_\xi\eta+(A+B_\xi)\eta_\xi)+O(\delta^4\bar{\eta}), \tag{3.16}$$

where $\bar{\eta}$ is a typical value of $\eta(\xi,t)$, since the term $-\delta^2B\eta_{\xi\xi}/h_0$ is of order $\delta^4\bar{\eta}$ from (3.15). Thus we may replace all derivatives of η in (3.14) by linear combinations of η and η_ξ. In general

$$\frac{\partial^{m+2}\eta}{\partial\xi^{m+2}} = -\frac{\delta^2}{h_0}\left[\frac{\partial^{m+2}A}{\partial\xi^{m+2}}\eta+\left\{(m+2)\frac{\partial^{m+1}A}{\partial\xi^{m+1}}+\frac{\partial^{m+2}B}{\partial\xi^{m+2}}\right\}\eta_\xi\right]. \tag{3.17}$$

This simplification has been obtained without making any assumptions about the length scales of variations in η, A and B except that they are $O(1)$ or larger. Substitution of (3.17) into (3.14) yields the equation

$$\begin{aligned}\frac{\delta^2\eta}{h_0}\left(-A+\frac{1}{3!}A_{\xi\xi}-\frac{1}{5!}A_{\xi\xi\xi\xi}\ldots+H-\frac{1}{2!}H_{\xi\xi}+\frac{1}{4!}H_{\xi\xi\xi\xi}\ldots\right)&\\ +\frac{\delta^2\eta_\xi}{h_0}\left(\frac{2}{3!}A_\xi-\frac{4}{5!}A_{\xi\xi\xi}\ldots-B+\frac{1}{3!}B_{\xi\xi}-\frac{1}{5!}B_{\xi\xi\xi\xi}\ldots\right.&\\ \left.-H_\xi+\frac{1}{3!}H_{\xi\xi\xi}-\frac{1}{5!}H_{\xi\xi\xi\xi\xi}\ldots\right) &= O(\delta^4\bar{\eta}).\end{aligned} \tag{3.18}$$

Thus if solutions of (3.15) are also to satisfy the bottom boundary condition to $O\{\delta^2\}$ we must have for consistency

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\partial^{2m} A}{\partial \xi^{2m}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{\partial^{2m} H}{\partial \xi^{2m}} \tag{3.19}$$

and

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\partial^{2m}}{\partial \xi^{2m}} (B + H_\xi) = \sum_{m=1}^{\infty} \frac{2m(-1)^{m+1}}{(2m+1)!} \frac{\partial^{2m-1} A}{\partial \xi^{2m-1}}. \tag{3.20}$$

Note that in (3.19) from the definition of H

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m!} \frac{\partial^{2m} H}{\partial \xi^{2m}} \equiv \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^{2m}}{2m!} \frac{\partial^{2m} x_\xi(\xi, 0)}{\partial \xi^{2m}} = x_\xi(\xi, -1) \tag{3.21}$$

from the structure of the conformal mapping. This expression will be singular at a stagnation point in the steady flow solution, however as will be shown, $A(\xi)$ is still an analytic function. Equation (3.19) may be solved by a combination of Green's function and complex variable methods (appendix B) to give

$$A(\xi) = \int_{-\infty}^{\infty} \frac{\pi}{4} \frac{x_\xi(\xi^*, -1)}{\cosh^2 \frac{1}{2}\pi(\xi - \xi^*)} d\xi^*. \tag{3.22}$$

$B(\xi)$ may then be obtained from (3.20) in the form

$$B(\xi) = \int_{-\infty}^{\infty} \frac{\pi}{4} \frac{(\xi^* - \xi)\, x_\xi(\xi^*, -1)}{\cosh^2 \frac{1}{2}\pi(\xi - \xi^*)} d\xi^* - H_\xi(\xi) \tag{3.23}$$

in close analogy with the expressions (3.4) and (3.5) for $a(\xi)$ and $b(\xi)$.

The numerical evaluation of (3.22) and (3.33) is considerably eased when there are singularities in $x_\xi(\xi, -1)$ if the transformation

$$x = x(\xi, -1) \tag{3.24}$$

is made to the physical co-ordinate x. Thus (3.22) takes the form

$$A(\xi^*) = \int_{-\infty}^{\infty} \frac{\pi}{2} \frac{dx}{\cosh^2 \frac{1}{2}\pi(\xi(x, -h(x)) - \xi^*)} \tag{3.25}$$

and the singularities caused by the presence of a stagnation point in the steady flow field are evanescent.

In figures 4 and 5 may be found graphs of A and C, where $C(\xi)$ is given by

$$B(\xi) = -\tfrac{2}{3} C_\xi(\xi), \quad C(-\infty) = h(-\infty), \tag{3.26}$$

and their approximations from (2.50) and (2.51): $H - \frac{1}{3}H_{\xi\xi}$ and H respectively. The topography considered in figure 4 is a step change in depth from 1 to R ($h_0 \equiv 1$), for which the conformal mapping takes the form

$$z(w) = \tanh^{-1}\chi - R\tanh^{-1}\chi,$$

where

$$\chi = \left\{\frac{R^2 + e^{\pi w}}{1 + e^{\pi w}}\right\}^{\frac{1}{2}}, \tag{3.27}$$

while that in figure 5 is a plane barrier of least depth R, for which

$$z(w) = (2/\pi) \sinh^{-1}(\sin \tfrac{1}{2}\pi R \sinh \tfrac{1}{2}\pi w). \tag{3.28}$$

For both cases we have taken $R = \frac{1}{2}$.

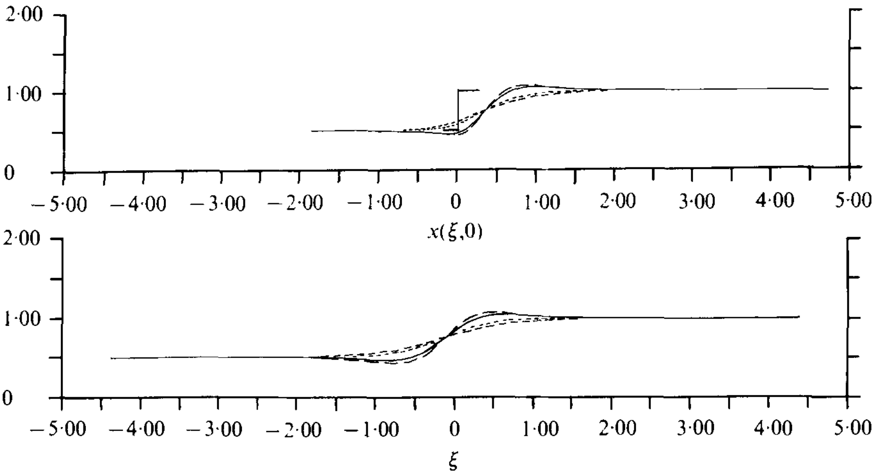

FIGURE 4. The functions A, C, $H-\frac{1}{3}H_{\xi\xi}$, H and $h(x(\xi, 0))$ for a step change in depth. ——, A; ---, C; — —, $H-\frac{1}{3}H_{\xi\xi}$; ······, H; ——, $h(x(\xi, 0))$.

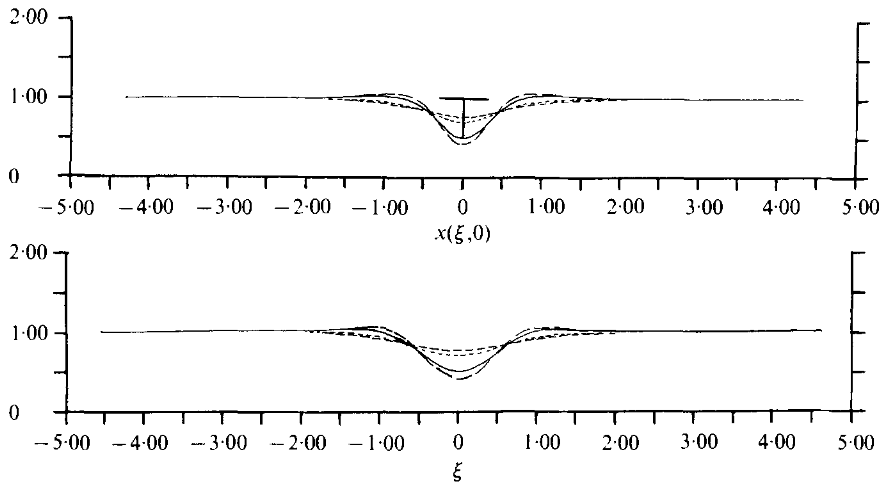

FIGURE 5. The functions A, C, $H-\frac{1}{3}H_{\xi\xi}$, H and $h(x(\xi, 0))$ for a half-depth spike. Curves as in figure 4.

Discussion of the long-period equation

In §2 we reformulated the long-wave equations in conformal co-ordinates and advanced arguments for the superiority of these equations over the conventional equations. The long-period equation, which is an added bonus of the conformal formulation, may now serve as a check on the accuracy of the reformulated equations. Thus the coefficients A and C can be compared with the coefficients $H-\frac{1}{3}H_{\xi\xi}$ and H respectively of (2.49) and in figures 4 and 5 we see that they are in reasonable agreement considering the extreme natures of the topographies considered. Conversely, the long-period equation cannot easily be extended to include effects of finite frequency (or amplitude); therefore in cases where we do not possess an exact solution for arbitrary frequencies and we wish to obtain some idea of the range of values of δ for which the long-period equation is accurate, it is useful to manufacture a correction for higher orders in δ by analogy with the reformulated equations. Figures 4 and 5 give us confidence that the reformulated equations will yield a correction at least of the correct order of magnitude.

Thus from (3.12), (3.13) and (2.16) we have

$$\eta_{\xi\xi}-\tfrac{1}{6}\eta_{\xi\xi\xi\xi}+(\delta^2/h_0)(H\eta-\tfrac{1}{2}(H\eta)_{\xi\xi}) = O\{\delta^4,\alpha^{-4}\} \tag{3.29}$$

and usnig the first-order balance

$$\eta_{\xi\xi}+(\delta^2/h_0)H\eta = O\{\delta^2,\alpha^{-2}\} \tag{3.30}$$

to remove the term in $\eta_{\xi\xi\xi\xi}$ we obtain

$$\eta_{\xi\xi}+(\delta^2/h_0)(H\eta-\tfrac{1}{3}(H\eta)_{\xi\xi}) = O\{\delta^4,\alpha^{-4}\}. \tag{3.31}$$

By analogy with (3.31) and using (2.50) and (2.51) we may now write down the equation

$$\eta_{\xi\xi}+(\delta^2/h_0)[(A+\tfrac{1}{3}C_{\xi\xi})\eta-\tfrac{1}{3}(C\eta)_{\xi\xi}] = O\{\delta^4,\delta^2\alpha^{-2}\}. \tag{3.32}$$

We term this equation the 'long-period equation with dispersion corrections'. It can be seen that $\eta_{\xi\xi} = O(\delta^2\eta)$ and therefore the long-period equation can be recovered from this equation merely by ignoring the term $-\tfrac{1}{3}(\delta^2/h_0)C\eta_{\xi\xi}$, which is $O(\delta^4\eta)$. We also remark that (3.32) and (3.31) in their time-dependent form possess rather better analytic properties than do (3.15) and (2.49). For example (3.31) may be written as a variational principle

$$\delta\iint_{-\infty}^{\infty} H\eta\mathscr{F}_t+\tfrac{1}{2}gH\eta^2+\tfrac{1}{2}\mathscr{F}_\xi^2-\tfrac{1}{6}H^2\eta_t^2\,d\xi\,dt = 0 \tag{3.33}$$

with associated equations

$$\delta\eta \rightarrow H\mathscr{F}_t+gH\eta+\tfrac{1}{3}H^2\eta_{tt} = 0, \tag{3.34}$$

$$\delta\mathscr{F} \rightarrow \eta_t+\mathscr{F}_{\xi\xi} = 0, \tag{3.35}$$

where $\mathscr{F}$ is approximately the average of the velocity potential over the depth (Hamilton 1974). However we cannot write down a variational principle for the time-dependent form of (2.49) since this equation is of odd order (Atherton & Hornsey 1975), and hence it is not possible to find a representation of the wave energy, which is identically conserved by this equation. However, from (3.33) we may obtain the equation (Eckart 1960)

$$\frac{\partial}{\partial t}\{\tfrac{1}{2}gH\eta^2+\tfrac{1}{2}\mathscr{F}_\xi^2+\tfrac{1}{6}H^2\eta_t^2\}-\frac{\partial}{\partial\xi}\{\mathscr{F}_t\mathscr{F}_\xi\} = 0. \tag{3.36}$$

Note that the energy density is positive definite, with important consequences for the stability of the solutions. For a fuller discussion of the merits of the various alternative forms of the higher-order terms in the long-wave equations (for constant depth only however) the reader is referred to the recent paper of Bona & Smith (1976).

In figures 6 and 7 we show values of the reflexion coefficients R_0 of a wave train incident on the topographies of figures 4 and 5 calculated from the long-period equation (3.15) and the long-period equation with dispersion corrections (3.32). Values calculated using the reformulated equations (2.49) and (3.31) were indistinguishable

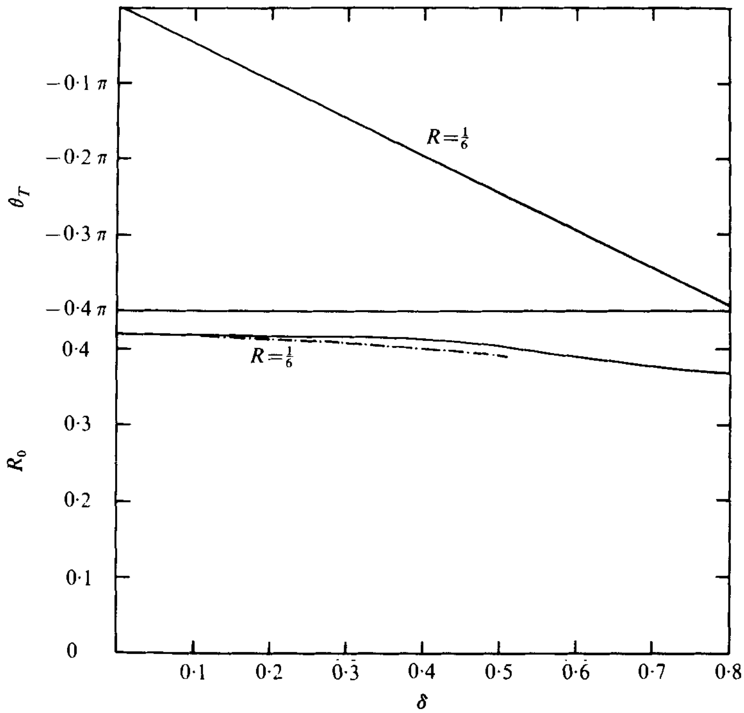

FIGURE 6. Reflexion coefficient and transmission phase angle for a step change in depth. —·—, long-period equation; ——, long-period equation with dispersion corrections.

within the accuracy of the plot from those calculated using (3.15) and (3.32) respectively. However, the reflexion coefficients are functions of the global properties of the equations and therefore this result should not obscure the essential differences between the exact and reformulated equations.

Also shown are the approximate theoretical results of Mei & Black (1969) for a submerged thin plate. These are known to be reasonably accurate (cf. Miles 1967) and are equally valid for all frequencies. The phase changes predicted here are indistinguishable from each other and from the results of Mei & Black within the accuracy of reading their graph. However, note that the phase speeds to which the phase changes are related are different for the different equations.

It can be seen that the reflexion coefficients predicted are in close agreement with each other and with the results of Mei & Black for frequencies up to $\delta = 0{\cdot}3$ ($\delta^2 \approx 0{\cdot}1$). This corresponds to a wavelength-to-depth ratio of about 20 and we suggest that on this evidence the long-period equation (3.15) will be adequate for most purposes up to this value.

As a further check we have calculated the reflexion coefficient for a wave incident on Roseau's (1952) topography, which is given by

$$z(w) = Rw + (1-R)\alpha^R \pi^{-1} \ln\left[\exp(\pi w/\alpha^R) + 1\right] \qquad (3.37)$$

(see figure 8). In this case comparison with Roseau's exact result would seem to suggest that there might be little quantitative advantage in using (3.32) in preference to (3.15).

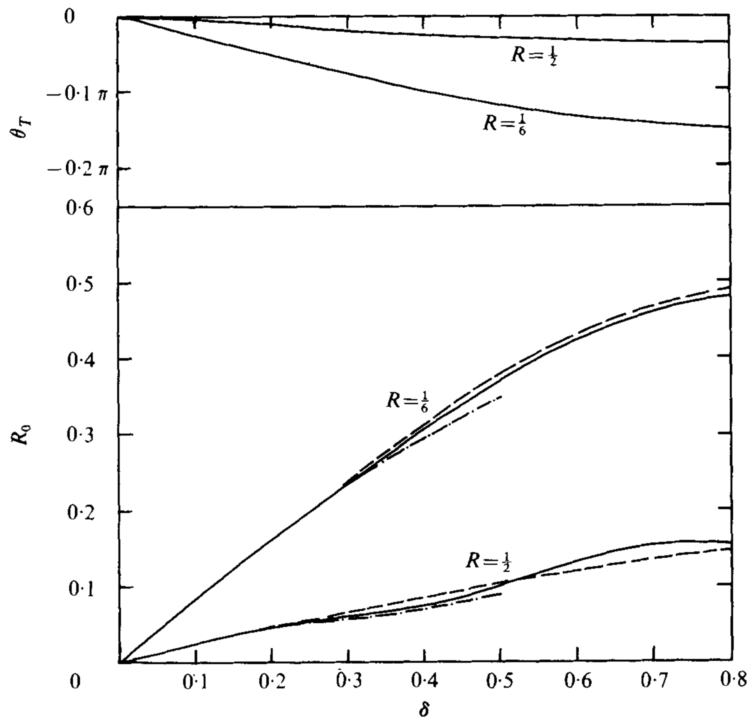

FIGURE 7. Reflexion coefficient and transmission phase angle for a submerged thin plate. ---, Mei & Black theory; —·—, long-period equation; ——, long-period equation with dispersion corrections.

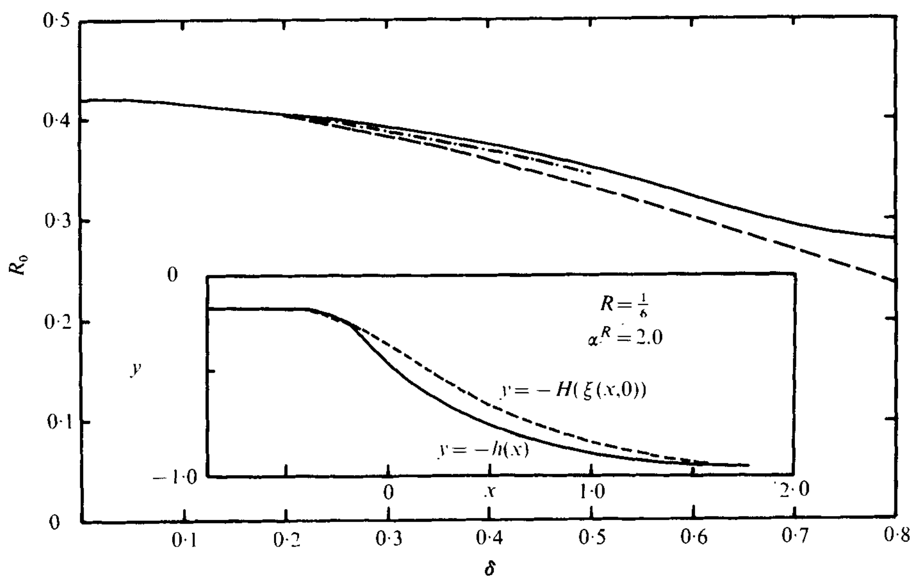

FIGURE 8. Reflexion coefficient for Roseau's topography. ---, Roseau (exact); —·—, long-period equation; ——, long-period equation with dispersion corrections.

4. Conclusions

In this paper we have developed equations of long-wave type based on an initial conformal mapping of the fluid domain (assumed two-dimensional) onto a strip which are aimed at improving the treatment of the effects of bottom topography without recourse to the more difficult short-wave theory. The basic equations (the 'reformulated long-wave equations') possess the advantages that the zeroth-order approximation to the fluid flow is much improved and that the Taylor series expansion on which the equations are based has improved convergence properties. In addition, in the limit of waves of infinitesimal amplitude and long period an exact equation has been derived and comparison of the coefficients indicates that the reformulated equations can be remarkably accurate for even the most abrupt bottom topographies.

I should like to acknowledge the support of N.E.R.C. with a research studentship whilst this work was being carried out and the help and advice of Dr P. A. Taylor of Southampton University and also of Dr D. H. Peregrine of Bristol University.

Appendix A. Nonlinear forms of the reformulated equation

Luke (1967) has shown that the governing equations for surface gravity waves may be obtained in the form of a variational principle. Thus in two dimensions the integral

$$I = \iiint_{-h(x,t)}^{\eta(x,t)} (\phi_t + gy + \tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2)\, dy\, dx\, dt \tag{A 1}$$

is stationary for small variations in ϕ and η. Note that the integrand is the pressure in the fluid. The principle is not affected by a change in co-ordinates, thus we may rewrite I in (ξ, ζ) co-ordinates as

$$I = \iiint_{-1}^{p(\xi,t)} \{J(\phi_t + gy) + \tfrac{1}{2}\phi_\xi^2 + \tfrac{1}{2}\phi_\zeta^2\}\, d\xi\, d\zeta\, dt, \tag{A 2}$$

where $\zeta = p(\xi, t)$ is the location of the free surface and

$$J(\xi,\zeta) = x_\xi y_\zeta - x_\zeta y_\xi = y_\xi^2 + y_\zeta^2 \tag{A 3}$$

is the Jacobian of the transformation (see Hamilton (1974) for $h = h(x,t)$ and hence $J = J(\xi,\zeta,t)$). The condition that I be stationary for variations δp in $p(\xi,t)$ and $\delta\phi$ in ϕ gives the equations

$$\left.\begin{aligned} &\delta p \to J(\phi_t + gy) + \tfrac{1}{2}\phi_\xi^2 + \tfrac{1}{2}\phi_\zeta^2 = 0 \quad &(\text{A } 4)\\ &\delta\phi \to Jp_t + \phi_\xi p_\xi - \phi_\zeta = 0 \quad &(\text{A } 5)\end{aligned}\right\} \text{ on } \zeta = p(\xi,t),$$

$$\phi_{\xi\xi} + \phi_{\zeta\zeta} = 0 \quad \text{in} \quad -1 < \zeta < p, \tag{A 6}$$

$$\phi_\zeta = 0 \quad \text{on} \quad \zeta = -1. \tag{A 7}$$

These equations are exact within the limits of inviscid irrotational theory, however it can be seen that for small amplitude waves (A 4) becomes

$$\phi_t + gHp = 0 \tag{A 8}$$

and (A 5)

$$H^2 p_t - \phi_\zeta = 0, \tag{A 9}$$

where $H(\xi) \equiv x_\xi(\xi, 0) \equiv y_\zeta(\xi, 0)$ as defined above and the wave amplitude

$$\eta \approx y_\zeta(\xi, 0)\, p = Hp. \tag{A 10}$$

Thus (A 8) and (A 9) are in agreement with (2.5) and (2.14) above.

Given (A 4)–(A 7) it is a straightforward matter to derive long-wave equations to any order by replacing ϕ by its Taylor expansion (2.21) [the solution of (A 6) and (A 7)] and substituting into (A 4) and (A 5) to obtain two equations for $p(\xi, t)$ and $f(\xi, t) = \phi(\xi, -1, t)$.

Note that approximations have to be made for the functions $J(\xi, p)$ and $Y(\xi, p)$. These may be obtained through (A 3) and the Taylor expansion for Y about $\zeta = 0$:

$$y(\xi, \zeta) = H\zeta - \frac{1}{3!} H_{\xi\xi}\zeta^3 + \frac{1}{5!} H_{\xi\xi\xi\xi}\zeta^5 \ldots, \tag{A 11}$$

this being the appropriate solution of Laplace's equation under the conditions

$$y = 0, \quad y_\zeta = H(\xi) \quad \text{on} \quad \zeta = 0. \tag{A 12), (A 13}$$

An extensive discussion of the types of equations which may be obtained is out of place here, however we quote the analogy with the Boussinesq (1871) equations containing first-order finite amplitude and dispersion effects (Hamilton 1974):

$$H^2\mathscr{F}_t + gH^3p + \tfrac{1}{3}H^4p_{tt} + \tfrac{1}{2}F_\xi^2 = O\{a^2, \alpha^{-4}\}, \tag{A 14}$$

$$H^2p_t + \partial[(1+p)\mathscr{F}_\xi]/\partial\xi = O\{a^2, \alpha^{-4}\}, \tag{A 15}$$

where

$$\mathscr{F}(\xi, t) = \int_{-1}^{0} \phi\, d\zeta \approx f - \tfrac{1}{3}f_{\xi\xi}$$

is the 'mean' value of the velocity potential over the depth and a is a typical amplitude-to-depth ratio of the wave. The variational formulation (A 2) makes the methods of Whitham (1967) appropriate and in fact (A 14) and (A 15) may be derived from the following variational principle by analogy with Whitham's equation for constant depth:

$$\delta \iint H^2p\mathscr{F}_t + \tfrac{1}{2}gH^3p^2 - \tfrac{1}{6}H^4p_t^2 + \tfrac{1}{2}(1+p)\mathscr{F}_\xi^2\, d\xi\, dt = 0. \tag{A 16}$$

This particular form of the long-wave equations has the advantage that the energy density conserved by the equations [the conservation equation of the principle (A 16)] is positive definite, namely

$$\frac{\partial}{\partial t}\{\tfrac{1}{2}gH^3p^2 + \tfrac{1}{2}(1+p)\mathscr{F}_\xi^2 + \tfrac{1}{6}H^4p_t^2\} - \frac{\partial}{\partial\xi}(1+p)\mathscr{F}_t\mathscr{F}_\xi = 0. \tag{A 17}$$

It is interesting that on eliminating $\mathscr{F}$ from linearized versions of (A 14) and (A 15) we obtain the equation

$$H^2p_{tt} - g\frac{\partial^2}{\partial\xi^2}(Hp) + \frac{\partial^2}{\partial\xi^2}[\tfrac{1}{3}H^2p_{tt}] = 0, \tag{A 18}$$

which is identical in form to the time-dependent version of the long-period equation with dispersion corrections (3.32):

$$\eta_{\xi\xi} + (\delta^2/h_0)[(A + \tfrac{1}{3}C_{\xi\xi})\eta - \tfrac{1}{3}(C\eta)_{\xi\xi}] = O(\delta^4, \delta^2\alpha^4) \tag{A 19}$$

(with $\eta = Hp$). Thus the nonlinear equations derivable from the variational principle

$$\delta\iint (A+\tfrac{1}{3}C_{\xi\xi})\,Hp\mathscr{F}_t+\tfrac{1}{2}g(A+\tfrac{1}{3}C_{\xi\xi})\,H^2p^2-\tfrac{1}{6}(A+\tfrac{1}{3}C_{\xi\xi})\,CH^2p_t^2+\tfrac{1}{2}(1+p)\,\mathscr{F}_\xi^2\,d\xi\,dt = 0 \tag{A 20}$$

possess the same formal justification as (A 14) and (A 15) since from (2.50) and (2.51)

$$A+\tfrac{1}{3}C_{\xi\xi} = H+O\{\alpha^{-4}\}$$

and

$$C = H+O\{\alpha^{-2}\}.$$

However, in the limit of small amplitude waves of long period they will in fact be exact rather than approximate as is the case for (A 14) and (A 15). The question of whether this increased accuracy in a particular limit is reflected in an overall increase in accuracy of the equations associated with (A 20) as compared with those associated with (A 16), however, cannot be answered on the information at present available.

Appendix B. Solution of (3.19) and (3.20) for $A(\xi)$ and $B(\xi)$

We consider first equation (3.19) for $A(\xi)$:

$$A-\frac{1}{3!}A_{\xi\xi}+\frac{1}{5!}A_{\xi\xi\xi\xi}\ldots = x_\xi(\xi,-1)$$
$$= \gamma(\xi),\quad \text{say}, \tag{B 1}$$

where $\gamma(\xi)$ is a function containing at the worst isolated singularities. We may invert this infinite-order differential equation by analogy with the properties of certain solutions of Laplace's equation. Thus we consider a function $\tilde{Y}(\xi,\zeta)$ which satisfies

$$\tilde{Y}_{\xi\xi}+\tilde{Y}_{\zeta\zeta} = 0 \quad \text{in} \quad -1<\zeta<1 \tag{B 2}$$

with boundary conditions

$$\tilde{Y}(\xi,\pm 1) = \pm\gamma(\xi). \tag{B 3}$$

Now since $\tilde{Y}$ is regular in $-1<\zeta<1$ and symmetrical about $\zeta = 0$ we may expand $\tilde{Y}$ in a Taylor expansion

$$\tilde{Y} = \zeta\gamma^*-\frac{\zeta^3}{3!}\gamma^*_{\xi\xi}+\frac{\zeta^5}{5!}\gamma^*_{\xi\xi\xi\xi}\ldots \tag{B 4}$$

which is convergent in $-1<\zeta<1$ and where

$$\gamma^*(\xi) = \tilde{Y}_\zeta(\xi,0) \tag{B 5}$$

is an analytic function of ξ. Thus from (B 3) we may identify

$$A(\xi) = \tilde{Y}_\zeta(\xi,0). \tag{B 6}$$

The fact that we do not have absolute convergence of the Taylor series on $\zeta = \pm 1$ if $\gamma(\xi)$ possesses singularities is not important since the Taylor series is subsidiary to the identification (B 6).

We may now obtain an inversion of (B 1) by using Green's theorem to obtain $\tilde{Y}_\zeta(\xi,0)$ from (B 2) and (B 3). Thus we define the complex Green's function

$$G(\xi,\zeta;\xi_0,\zeta_0) = \log\left(\frac{\exp(\frac{1}{2}\pi w)-\exp(\frac{1}{2}\pi w_0)}{\exp(\frac{1}{2}\pi w)-\exp(\frac{1}{2}\pi\bar{w}_0)}\right), \tag{B 7}$$

where $w = \xi + i\zeta$, $w_0 = \xi_0 + i\zeta_0$, the overbar denotes the complex conjugate and G satisfies

$$G \to \log(w - w_0) \quad \text{near} \quad w = w_0, \tag{B 8}$$

$$\operatorname{Re} G \equiv 0 \quad \text{on} \quad \zeta = \pm 1 \tag{B 9}$$

and

$$\operatorname{Re} G, \operatorname{Re} G_\xi \to 0 \quad \text{as} \quad \xi \to \pm\infty \quad \text{in} \quad -1 \leqslant \zeta \leqslant 1. \tag{B 10}$$

Thus using Green's theorem we have

$$2\pi \tilde{Y}(\xi_0, \zeta_0) = \operatorname{Re} \int_C \tilde{Y} G_n \, ds, \tag{B 11}$$

where C is the circuit around the strip $-1 \leqslant \zeta \leqslant 1$ and $\hat{\mathbf{n}}$ is the unit outward normal.

Thus

$$2\pi \tilde{Y}(\xi_0, \zeta_0) = \operatorname{Re} \int_{-\infty}^{\infty} \gamma(\xi) (G_{\zeta, \zeta=+1} + G_{\zeta, \zeta=-1}) \, d\xi \tag{B 12}$$

and hence from (B 6)

$$A(\xi_0) = \operatorname{Re} \int_{-\infty}^{\infty} \frac{\gamma(\xi)}{2\pi} (G_{\zeta\zeta_0, \zeta=+1} + G_{\zeta\zeta_0, \zeta=-1}) \, d\xi. \tag{B 13}$$

After some simplification we obtain

$$A(\xi_0) = \int_{-\infty}^{\infty} \gamma(\xi) \frac{\frac{1}{4}\pi}{\cosh^2[\frac{1}{2}\pi(\xi - \xi_0)]} \, d\xi. \tag{B 14}$$

Notice that

$$\int_{-\infty}^{\infty} \frac{\pi}{4} \frac{d\xi}{\cosh^2[\frac{1}{2}\pi(\xi - \xi_0)]} = [\tfrac{1}{2} \tanh[\tfrac{1}{2}\pi(\xi - \xi_0)]]_{-\infty}^{\infty} = 1,$$

thus $A(\xi_0)$ is a smoothed local 'average' of $\gamma(\xi_0)$. Also, as noted above, if we change to the x co-ordinate we may write

$$A(\xi_0) = \int_{-\infty}^{\infty} \frac{\pi}{4} \frac{dx}{\cosh^2[\frac{1}{2}\pi(\xi(x, -h(x)) - \xi_0)]} \tag{B 15}$$

and $A(\xi)$ is not sensitive to the presence of singularities in $x_\xi(\xi, -1) = \gamma(\xi)$ (i.e. corners in the depth profile $y = -h(x)$).

We may follow essentially the same procedure to find $B(\xi)$, however with more labour. Thus equation (3.20) for B takes the form

$$\mathscr{L}(B + H_\xi) = \frac{2}{3!} A_\xi - \frac{4}{5!} A_{\xi\xi\xi} \cdots, \tag{B 16}$$

where $\mathscr{L}$ is the differential operator which we have already inverted:

$$\mathscr{L} = 1 - \frac{1}{3!} \frac{\partial^2}{\partial \xi^2} + \frac{1}{5!} \frac{\partial^4}{\partial \xi^4} \cdots. \tag{B 17}$$

Now we may write the right-hand side of (B 16) in the form

$$\left[\frac{1}{2!} \tilde{X}_{\xi\xi} - \frac{1}{4!} \tilde{X}_{\xi\xi\xi\xi} \cdots - \frac{1}{3!} \tilde{X}_{\xi\xi} + \frac{1}{5!} \tilde{X}_{\xi\xi\xi\xi} \cdots \right]_{\zeta=0}, \tag{B 18}$$

where $\tilde{X}(\xi, \zeta)$ is the real function associated with $\tilde{Y}(\xi, \zeta)$ such that

$$\tilde{X} + i\tilde{Y} = \tilde{Z}(\xi + i\zeta) \tag{B 19}$$

and may be thought of as being defined by the Cauchy–Riemann relations

$$\tilde{X}_\xi = \tilde{Y}_\zeta, \quad \tilde{X}_\zeta = -\tilde{Y}_\xi. \tag{B 20), (B 21}$$

Thus $\tilde{X}(\xi,\zeta)$ has the Taylor expansion

$$\tilde{X}(\xi,\zeta) = \tilde{X}(\xi,0) - \frac{\zeta^2}{2!}\tilde{X}_{\xi\xi}(\xi,0)\ldots. \tag{B 22}$$

Note that $\tilde{X}$ is only defined to within an additive constant. Now (B 16) can be re-written in the form

$$\mathscr{L}(B+H_\xi) = \lim_{\zeta^*\to+1} \tfrac{1}{2}\int_{-\zeta^*}^{\zeta^*} [\tilde{X}(\xi,\zeta_0) - \tilde{X}(\xi,-\zeta^*)]\,d\zeta_0. \tag{B 23}$$

We need to consider the limit $\zeta^*\to 1$ in this manner to avoid difficulties in the application of (B 11) when $w = w_0$ is on the boundary.

Now from (B 12) and (B 21)

$$2\pi\tilde{X}_{\zeta_0}(\xi_0,\zeta_0) = \mathrm{Re}\left\{-\int_{-\infty}^{\infty}\gamma(\xi)\,[G_{\zeta\xi_0}]_{\zeta=\pm1}d\xi\right\}, \tag{B 24}$$

where $\zeta = \pm 1$ is taken to mean that we add together the values of the quantity at $\zeta = +1$ and $\zeta = -1$. Thus, using this notation

$$2\pi\{\tilde{X}(\xi_0,\zeta_0) - \tilde{X}(\xi_0,-\zeta^*\} = \mathrm{Re}\int_{-\infty}^{\infty} -\gamma(\xi)\left(\int_{-\zeta^*}^{\zeta_0} G_{\zeta\xi_0}d\zeta_0\right)_{\zeta=\pm1} d\xi. \tag{B 25}$$

Now

$$G_\zeta = \frac{i\pi}{2}\frac{\exp(\frac{1}{2}\pi w)}{\exp(\frac{1}{2}\pi w) - \exp(\frac{1}{2}\pi w_0)} - \frac{i\pi}{2}\frac{\exp(\frac{1}{2}\pi w)}{\exp(\frac{1}{2}\pi w) + \exp(\frac{1}{2}\pi \overline{w}_0)},$$

so that

$$G_{\zeta\xi_0} = i\left(\frac{\pi}{2}\right)^2\left[\frac{\exp[\frac{1}{2}\pi(w-w_0)]}{[\exp(\frac{1}{2}\pi w) - \exp(\frac{1}{2}\pi w_0)]^2} + \frac{\exp[\frac{1}{2}\pi(w-\overline{w}_0)]}{[\exp(\frac{1}{2}\pi w) + \exp(\frac{1}{2}\pi \overline{w}_0)]^2}\right]$$

and

$$\int_{-\zeta^*}^{\zeta_0} G_{\zeta\xi_0}d\zeta_0 = \frac{\pi}{2}\left[\frac{\exp(\frac{1}{2}\pi w)}{\exp(\frac{1}{2}\pi w) - \exp(\frac{1}{2}\pi w_0)} + \frac{\exp(\frac{1}{2}\pi w)}{\exp(\frac{1}{2}\pi w) + \exp(\frac{1}{2}\pi \overline{w}_0)}\right]_{w_0=\xi_0-i\zeta^*}^{w_0=\xi_0+i\zeta_0}. \tag{B 26}$$

Note that on $\zeta = \pm 1$ the second term is the complex conjugate of the first, so that we may drop 'Re' from (B 25). Thus

$$\begin{aligned} 2\pi\int_{-\zeta^*}^{\zeta^*} &(\tilde{X}(\xi,\zeta_0) - \tilde{X}(\xi_0,\zeta^*)\,d\zeta_0 \\ &= \int_{-\infty}^{\infty}\gamma(\xi)\left[\frac{\pi\zeta^*}{1-\exp[\frac{1}{2}\pi(\xi_0-w-i\zeta^*)]} + \frac{\pi\zeta^*}{1+\exp[\frac{1}{2}\pi(\xi_0-w+i\zeta^*)]}\right. \\ &\quad \left.+ i\log\left[\frac{1-\exp[\pi(w-\xi_0+i\zeta^*)]}{1-\exp[\pi(w-\xi_0-i\zeta^*)]}\right]\right]_{\zeta=\pm1} d\xi \end{aligned} \tag{B 27}$$

and hence

$$\begin{aligned} (B+H_\xi)_{\xi=\xi^*} = \lim_{\zeta^*\to1}\int_{-\infty}^{\infty}&\frac{\pi}{4}\frac{1}{\cosh^2[\frac{1}{2}\pi(\xi^*-\xi_0)]}\int_{-\infty}^{\infty}\gamma(\xi)\left[\frac{\frac{1}{4}\zeta^*}{1-\exp[\frac{1}{2}\pi(\xi_0-w-i\zeta^*)]}\right. \\ &\left.+\frac{\frac{1}{4}\zeta^*}{1+\exp[\frac{1}{2}\pi(\xi_0-w+i\zeta^*)]} + \frac{i}{4\pi}\log\left\{\frac{1-\exp[\pi(w-\xi_0+i\zeta^*)]}{1-\exp[\pi(w-\xi_0-i\zeta^*)]}\right\}_{\zeta=\pm1}\right]d\xi\,d\xi_0 \end{aligned} \tag{B 28}$$

$$= \lim_{\zeta^*\to1}\int_{-\infty}^{\infty}\gamma(\xi)\,K(\xi,\xi^*,\zeta^*)\,d\xi, \tag{B 29}$$

say, where $K(\xi, \xi^*, \zeta^*)$ can be written as the contour integral

$$K(\xi,\xi^*,\zeta^*) = \mathrm{Im}\oint_C \frac{\pi}{4}\frac{(w'+\alpha)}{\sinh^2[\frac{1}{2}\pi(\xi^*-\xi-w')]}\left[\frac{\frac{1}{4}\zeta^*}{1-\exp[\frac{1}{2}\pi(w'-i\zeta^*)]}\right.$$
$$\left.+\frac{\frac{1}{4}\zeta^*}{1+\exp[\pi(w'+i\zeta^*)]}+\frac{i}{4\pi}\log\left(\frac{1-\exp[\pi(w'+i\zeta^*)]}{1-\exp[\pi(w'-i\zeta^*)]}\right)\right]dw', \quad \text{(B 30)}$$

where we have defined

$$w' = \xi_0 - w \quad \text{(B 31)}$$

such that the circuit C is around the domain $-1 \leqslant \mathrm{Im}\, w' \leqslant 1$, $\mathrm{Im}\, w'$ taking the values ± 1 on C. Thus

$$\sinh^2[\tfrac{1}{2}\pi(\xi^*-\xi-w')] = -\cosh^2[\tfrac{1}{2}\pi(\xi^*-\xi_0)] \quad \text{on} \quad w' = \xi_0 - \xi \pm i \quad \text{(B 32)}$$

and

$$\mathrm{Im}\left\{\frac{w'+\alpha}{\sinh^2[\frac{1}{2}\pi(\xi^*-\xi-w')]}\right\} = \mp\frac{1}{\cosh^2[\frac{1}{2}\pi(\xi^*-\xi_0)]} \quad \text{on} \quad \mathrm{Im}\, w' = \pm i, \quad \text{(B 33)}$$

where α is a real constant (for convenience not taken equal to $\xi - \xi^*$).

Now the singularities of (B 30) within $-1 \leqslant \mathrm{Im}\, w' \leqslant 1$ consist of a second-order pole at $w' = \xi^* - \xi$, a first-order pole at $w' = i\zeta^*$, and logarithmic singularities at $w' = \pm i\zeta^*$. The complications introduced by the logarithmic singularities can be avoided if in these cases we take $\lim \zeta^* \to 1$ before the integration around the circuit. The log term then becomes

$$\frac{i}{4\pi}\log\left[\frac{1+\exp[\pi(w-\xi_0)]}{1+\exp[\pi(w-\xi_0)]}\right] \equiv 0. \quad \text{(B 34)}$$

Thus only the first- and second-order poles contribute to $K(\xi, \xi^*, \zeta^*)$ in the limit $\zeta^* \to 1$. Hence, neglecting the logarithmic contribution,

$$K(\xi,\xi^*,\zeta^*) = \mathrm{Im}\, 2\pi i \left\{\frac{\pi}{4}\frac{(\xi^*-\xi+\alpha)}{(\frac{1}{2}\pi)^2}\right.$$
$$\times\left[\frac{1}{4}\zeta^*\frac{\pi}{2}\frac{\exp[\frac{1}{2}\pi(\xi^*-\xi-i\zeta^*)]}{\{\exp[\frac{1}{2}\pi(\xi^*-\xi-i\zeta^*)]-1\}^2}-\frac{1}{4}\zeta^*\frac{\pi}{2}\frac{\exp[\frac{1}{2}\pi(\xi^*-\xi+i\zeta^*)]}{\{\exp[\frac{1}{2}\pi(\xi^*-\xi+i\zeta^*)]+1\}^2}\right]$$
$$\left.-\frac{\pi}{4}\frac{(\alpha+i\zeta^*)}{\sinh^2[\frac{1}{2}\pi(\xi^*-\xi-i\zeta^*)]}\left[\frac{1}{4}\frac{\zeta^*}{\frac{1}{2}\pi}\right]\right\} \quad \text{(B 35)}$$

and taking the limit $\zeta^* \to 1$ gives

$$(B+H_\xi)_{\xi=\xi^*} = \int_{-\infty}^{\infty}\gamma(\xi)\left[2(\xi^*-\xi+\alpha)\,\mathrm{Re}\left\{-\frac{\pi}{4}\frac{i\exp[\frac{1}{2}\pi(\xi^*-\xi)]}{\{1+i\exp[\frac{1}{2}\pi(\xi^*-\xi)]\}^2}\right\}\right.$$
$$\left.+\frac{\pi}{4}\frac{\alpha}{\cosh^2[\frac{1}{2}\pi(\xi^*-\xi)]}\right]d\xi, \quad \text{(B 36)}$$

from which

$$B(\xi^*) = \int_{-\infty}^{\infty}\gamma(\xi)\frac{\pi}{4}\frac{(\xi-\xi^*)}{\cosh^2[\frac{1}{2}\pi(\xi-\xi^*)]}d\xi - H_\xi(\xi^*), \quad \text{(B 37)}$$

or on making the transformation $x = x(\xi, -1)$,

$$B(\xi^*) = \int_{-\infty}^{\infty}\frac{\pi}{4}\frac{(\xi(x,-h(x))-\xi^*)}{\cosh^2[\frac{1}{2}\pi(\xi(x,-h(x))-\xi^*)}dx - H_\xi(\xi^*) \quad \text{(B 38)}$$

in agreement with (3.23). Note that perturbations in the bottom topography which have a small effect on the value of the potential function $\xi(x, y)$ on the bottom, or alternatively small shifts in the location of the bottom of a line of constant ξ compared with the depth will also have a small effect on the values of $A(\xi)$ and $B(\xi)$. The long-period equation is thus in some sense stable with respect to perturbations of this kind.

REFERENCES

ATHERTON, R. W. & HORNSY, G. M. 1975 On the existence and formulation of variational principles for non linear differential equations. *Stud. Appl. Math.* **54**, 31–59.

BARTHOMLOEUSZ, E. F. 1958 The reflection of long waves at a step. *Proc. Camb. Phil. Soc.* **54**, 106–118.

BONA, J. L. & SMITH, R. 1976 A model for the two way propagation of water waves in a channel. *Math. Proc. Camb. Phil. Soc.* **79**, 167–182.

BOUSSINESQ, M. J. 1871 Théorie générale des mouvements qui son propagés dans un canal rectangulaire horizontales. *C. R. Acad. Sci. Paris* **73**, 256–260.

ECKART, C. 1960 Variational problems of hydrodynamics. *Phys. Fluids* **3**, 421–427.

HAMILTON, J. 1974 On the effects of depth variation on long waves. Ph.D. thesis, University of Southampton.

KREISEL, G. 1949 Surface waves. *Quart. Appl. Math.* **7**, 21–44.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LUKE, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

MEI, C. C. & BLACK, J. L. 1969 Scattering of surface waves by rectangular obstacles in water of finite depth. *J. Fluid Mech.* **38**, 499–511.

MEI, C. C. & LE MEHAUTÉ, B. L 1966 Note on the equation of long waves over an uneven bottom. *J. Geophys. Res.* **71**, 393–400.

MILES, J. W. 1967 Surface wave scattering matrix for a shelf. *J. Fluid Mech.* **28**, 755–769.

PEREGRINE, D. H. 1967 Long waves on a beach. *J. Fluid Mech.* **27**, 815–827.

PHILLIPS, E. G. 1957 Functions of a complex variable with applications. *Univ. Math. Texts. Lond.*

ROSEAU, M. 1952 Contributions à la théorie des ondes liquides de gravité en profondeur variable. *Pub. Sci. Tech. Min. Air* no. 1.

WHITHAM, G. B. 1967 Variational methods and application to water waves. *Proc. Roy. Soc. A.* **299**, 6–25.